Topological recursion, quantum curves and the second Painleve equation (Microlocal Analysis and Singular Perturbation Theory)

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By

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Abstract

For the second Painlevé equation P_2, we show that the WKB solution of the Jimbo-Miwa isomonodromy system (with the 0-parameter solution of P_2 being substituted) is obtained by the Eynard-Orantin topological recursion from the spectral curve of P_2.

§ 1. Introduction

Since the discovery of *Painlevé equations* by [28], a lot of works have been devoted to investigate properties of solutions of Painlevé equations (e.g., [12]). One remarkable property of the Painlevé equations is the existence of the associated *isomonodromy system* (or the *Lax pair*). That is, each Painlevé equation describes an isomonodromic deformation of a certain meromorphic linear ordinary differential equation ([20, 21]). Based on the theory of *exact WKB analysis* of the isomonodromy system, Aoki, Kawai and Takei established the WKB theoretic approach to Painlevé equations with a large parameter η ($= 1/\hbar$ in this article). See [1, 23, 24, 25, 29]. The second Painlevé equation, which we will discuss in this article, was studied in this framework by [14, 15, 16] in detail.

On the other hand, Eynard-Orantin's *topological recursion* ([11]) is a recursive algorithm to assign a hierarchy of certain invariants (or functions) to a given algebraic curve called a *spectral curve.* Topological recursion attracts both mathematicians and physicists since these invariants are expected to encode the information of various geometric or enumerative invariants and correlation functions of matrix models (see [10]).

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Moreover, the topological recursion is also closely related to integrable systems and the theory of Painlevé equations (see [3, 4, 17, 18]).

For a class of spectral curves, it turns out that the generating function of invariants defined by the topological recursion satisfies a Schrödinger-type linear differential (or difference) equation containing a small parameter $\hbar$ (e.g., [2, 5, 6, 7, 8, 9, 13, 26]). The Schrödinger-type equation is said to be a *quantization* of the spectral curve, or *quantum curve*, because its *semi-classical limit* $\hbar \to 0$ recovers the initial spectral curve. As an application, the WKB analysis of quantum curves enable us to describe certain formulas for Gromov-Witten invariants ([8, 9]). Understanding how Schrödinger-type equations solve enumerative problems etc. is very interesting problem, but we haven't established a general theory so far. The above literatures on quantum curves (as well as this article) will be foundation to build a general theory of quantum curves.

Here let us focus on the connection between the Painlevé equations and quantum curves. A construction of the isomonodromy system as a quantum curve is first established in [19] for the case of the first Painlevé equation with a small parameter $\hbar$. The main result of this article is to generalize the result to the second Painlevé equation

$$P_2 : \ \hbar^2 \frac{d^2 q}{dt^2} = 2q^3 + tq - \theta + \frac{\hbar}{2}.$$

Here θ is a complex parameter. A new observation of this article is that we have obtained two different quantum curves from the same topological recursion process. That is, the quantum curve depends on how we integrate to get a WKB solution defined through the topological recursion, as we will describe below.

Now we briefly describe our main results. Let us take

$$y^2 = (x - q_0)^2 (x^2 + 2q_0 x + 3q_0^2 + t) \tag{1.1}$$

as a (1-parameter family of) spectral curve for the topological recursion. Here $q_0 = q_0(t)$ the leading term of $\hbar$-expansion of the solution of P_2. Let $W_{g,n}(z_1, \dots, z_n)$ be the *Eynard-Orantin differential* of type (g, n) defined from the spectral curve (1.1) (see Section 3.1). These are meromorphic differential forms (and depend also on t), where z_i's are copies of a coordinate on the spectral curve (1.1). Let us introduce functions

$$F_{g,n}(z_1, \dots, z_n) = \int_{\infty}^{z_1} \cdots \int_{\infty}^{z_n} W_{g,n}(z_1, \dots, z_n) \tag{1.2}$$

satisfying $d_{z_1} \cdots d_{z_n} F_{g,n} = W_{g,n}$. Then, our main results claims that a particular WKB solution of the isomonodromy system is constructed as the generating function of $F_{g,n}$ (after appropriate modifications of $F_{0,1}$ and $F_{0,2}$).

Theorem 1.1 (Theorem 3.2). *The formal series $\psi(x, t, \hbar)$ defined by*

$$\psi(x, t, \hbar) = \exp\left(\sum_{g \geq 0, n \geq 1} \hbar^{2g-2+n} \left. \frac{F_{g,n}(z, \dots, z)}{n!} \right|_{z = z(x)} \right)$$

is a WKB solution of an isomonodromy system (2.7) *and* (2.8) *associated with* P_2. *Here* $z(x)$ *is the inverse function of* $x(z)$ *which parametrizes the spectral curve* (1.1). *The semi-classical limit of the isomonodromy system coincides with the spectral curve* (1.1).

Here we show another result. Let us introduce

$$\bar{F}_{g,n}(z_1, \ldots, z_n) = \frac{1}{2^n} \int_{\bar{z}_1}^{z_1} \cdots \int_{\bar{z}_n}^{z_n} W_{g,n}(z_1, \ldots, z_n) \tag{1.3}$$

(where $\bar{z} = 1/z$) which also satisfies $d_{z_1} \cdots d_{z_n} \bar{F}_{g,n} = W_{g,n}$ as well as $F_{g,n}$. Our second main result claims that the corresponding quantum curve is also related to the second Painlevé equation, but different from the one in Theorem 1.1.

Theorem 1.2 (Theorem 5.1). *The formal series* $\bar{\psi}(x, t, \hbar)$ *defined by*

$$\bar{\psi}(x, t, \hbar) = \exp\left(\sum_{g \geq 0, n \geq 1} \hbar^{2g-2+n} \frac{\bar{F}_{g,n}(z, \ldots, z)}{n!} \bigg|_{z=z(x)} - a(t, \hbar) \right)$$

is a WKB solution of an isomonodromy system (5.7) *and* (5.8) *associated with*

$$\bar{P}_2 : \ \hbar^2 \frac{d^2 q}{dt^2} = 2q^3 + tq - \theta.$$

Here $a(t, \hbar)$ *is a certain formal series independent of* x. *The semi-classical limit of the isomonodromy system coincides with the spectral curve* (1.1).

Therefore the quantum curve is again an isomonodormy system of the second Painlevé equation, but with a parameter shift $\theta \mapsto \theta + \hbar/2$. These results show that the quantum curve depends on how we define the function $F_{g,n}$ satisfying $d_{z_1} \cdots d_{z_n} F_{g,n} = W_{g,n}$. This is our new observation which cannot be seen in the case of the first Painlevé equation. This is because the spectral curve of the first Painlevé equation has only one ramification point, and hence $F_{g,n}$ and $\bar{F}_{g,n}$ coincide (see [19, Remark 4.8]).

This article is organized as follows. In Section 2, we briefly review some known facts about P_2 and the WKB analysis of the isomonodromy system [15, 23]. Theorem 1.1 (Theorem 3.2) will be formulated precisely in Section 3 after recalling the notion of topological recursion. We will give a proof of Theorem 1.1 (= Theorem 3.2) and Theorem 1.2 (= Theorem 5.1) in Section 4 and 5, respectively.

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§ 2. The second Painlevé equation P_2

Painlevé equations having a small parameter $\hbar$ (or a large parameter $1/\hbar$) has been analyzed by Aoki, Kawai and Takei in terms of the exact WKB analysis ([1, 23, 24, 25]). In this article we focus on the second Painlevé equation with a small parameter $\hbar$:

$$P_2 \ : \ \hbar^2 \frac{d^2 q}{dt^2} = 2q^3 + tq - \theta + \frac{\hbar}{2}.$$

The equation P_2 is analyzed by [14, 15] in detail. We review some results of these literatures that are relevant to this article.

§ 2.1. Hamiltonian description and formal power series solution

It is well-known that P_2 can be written as a Hamiltonian system with the (time-depending) Hamiltonian

$$H = H(q,p,t) = \frac{1}{2}p^2 + \left(q^2 + \frac{t}{2}\right)p + \theta q. \tag{2.1}$$

That is, P_2 is equivalent to

$$\begin{cases} \hbar \dfrac{dq}{dt} = \dfrac{\partial H}{\partial p} = p + q^2 + \dfrac{t}{2}, \\ \hbar \dfrac{dp}{dt} = -\dfrac{\partial H}{\partial q} = -2qp - \theta. \end{cases} \tag{2.2}$$

(See [27].) Throughout of the paper we only focus on the formal power series solution (called *0-parameter solution*) of (2.2) of the form

$$(q,p) = (q(t,\hbar), p(t,\hbar)) = \Big(\sum_{n=0}^{\infty} \hbar^n q_n(t), \sum_{n=0}^{\infty} \hbar^n p_n(t)\Big). \tag{2.3}$$

It is easy to see that $q(t,\hbar)$ is the formal power series solution of P_2. Hence the top term $q_0 = q_0(t)$ satisfies an algebraic equation

$$2q_0^3 + tq_0 - \theta = 0, \tag{2.4}$$

and the terms $q_n(t)$ for $n \geq 1$ are recursively determined. The formal power series $p(t,\hbar)$ is given by

$$p(t,\hbar) = \hbar \frac{dq}{dt}(t,\hbar) - q(t,\hbar)^2 - \frac{t}{2}.$$

In what follows we assume

$$\theta \neq 0. \tag{2.5}$$

Note that q_0 never vanishes under the assumption (2.5). We also assume that t lies on a domain defined by

$$\left(6q_0(t)^2 + t =\right) \; 4q_0(t)^2 - \frac{\theta}{q_0(t)} \neq 0. \tag{2.6}$$

Remark 2.1. The points excluded in (2.6) are called *turning points* of P_2. Turning points, together with *Stokes curves* introduced in [23] play important role when we discuss the *Borel summability* ([22]) and the *(parametric) Stokes phenomenon* ([14, 15, 16]) of the formal solution of P_2.

§ 2.2. Isomonodromy system

It is well-known that each Painlevé equation describes the compatibility condition of a certain system of linear PDEs, called *isomonodormy system* ([20, 21]). For P_2,

$$\hbar^2 \frac{\partial^2 \psi}{\partial x^2} = \hbar f \frac{\partial \psi}{\partial x} + g\psi, \tag{2.7}$$

$$\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar}{2(x-q)} \frac{\partial \psi}{\partial x} - \Big(\frac{p + q^2 + t/2}{2(x-q)} - \frac{q}{2} \Big) \psi \tag{2.8}$$

give an isomonodromy system. Here

$$f = \hbar \frac{1}{x-q}, \quad g = x^4 + tx^2 - 2\theta x + 2H + \frac{t^2}{4} + \hbar(x-q) - \hbar \frac{p + q^2 + t/2}{x-q},$$

and H is given in (2.1). In what follows we consider the system (2.7) and (2.8) with the formal solution (2.3) being substituted into the coefficients. Therefore f and g are regarded as a formal power series $f = \sum_{n=0}^{\infty} \hbar^n f_n$, $g = \sum_{n=0}^{\infty} \hbar^n g_n$ whose top terms are given by

$$f_0 = 0, \quad g_0 = (x - q_0)^2 (x^2 + 2q_0 x + 3q_0^2 + t).$$

Remark 2.2. The isomonodromy system (2.7) and (2.8) is obtained from the following matrix-version of the isomonodromy system introduced by Jimbo and Miwa (for the case of $\hbar = 1$) in [20, Appendix C]:

$$\hbar \frac{\partial \Psi}{\partial x} = A\Psi, \quad \hbar \frac{\partial \Psi}{\partial t} = B\Psi,$$

where

$$A = \begin{pmatrix} x^2 + p + \dfrac{t}{2} & u(x-q) \\ -\dfrac{2(xp + qp + \theta)}{u} & -x^2 - p - \dfrac{t}{2} \end{pmatrix}, \qquad B = \begin{pmatrix} \dfrac{x}{2} & \dfrac{u}{2} \\ -\dfrac{p}{u} & -\dfrac{x}{2} \end{pmatrix}.$$

Here q and p are solution of (2.2), and u satisfies $\hbar(du/dt) = -q$. It is easy to check that, if we write $\Psi = {}^t(\psi, \varphi)$, then the first component ψ satisfies

$$\hbar^2 \frac{\partial^2 \psi}{\partial x^2} = \hbar^2 \frac{1}{A_{12}} \frac{\partial A_{12}}{\partial x} \frac{\partial \psi}{\partial x} + \left(-\det A + \hbar \frac{\partial A_{11}}{\partial x} - \hbar \frac{A_{11}}{A_{12}} \frac{\partial A_{12}}{\partial x} \right) \psi,$$
$$\hbar \frac{\partial \psi}{\partial t} = \hbar \frac{B_{12}}{A_{12}} \frac{\partial \psi}{\partial x} + \left(B_{11} - \frac{A_{11} B_{12}}{A_{12}} \right) \psi.$$

A straightforward computation shows that these equations coincide with (2.7) and (2.8), respectively.

Remark 2.3. The system (2.7) and (2.8) is obtained from the isomonodromy system considered in [15, 23] through the gauge transformation $\psi \mapsto u^{-1/2}(x-q)^{1/2}\psi$ together with a shift $\theta \mapsto \theta - \hbar/2$ of the parameter. Hence, results proved in the references are applicable to our current situation with some modification.

§ 2.3. τ-function of P_2

Let

$$\sigma(t, \hbar) = H(q(t, \hbar), p(t, \hbar), t) \tag{2.9}$$

be the Hamiltonian function. The formal series $\tau(t, \hbar)$ satisfying

$$\hbar^2 \frac{d^2}{dt^2} \log \tau(t, \hbar) = \sigma(t, \hbar) \tag{2.10}$$

is called the *(isomonodromic) τ-function* of P_2 ([21, 27]). As is shown in [17, Section 3.2], the τ-function for P_2 is invariant under $\hbar \mapsto -\hbar$. Hence, the formal series σ is a formal power series of the form

$$\sigma(t, \hbar) = \sum_{n=0}^{\infty} \hbar^{2n} \sigma_{2n}(t). \tag{2.11}$$

§ 2.4. WKB solution of the isomonodoromy system

Let us summarize some part of results obtained in [15, 23] which is relevant to the main result of this article.

The WKB solution is a formal power series solution of the form

$$\psi_{\pm}(x, t, \hbar) = \exp\left(\int^x R^{(\pm)}(x, t, \hbar) dx \right), \tag{2.12}$$

where $R^{(\pm)}(x, t, \hbar) = \sum_{m=0}^{\infty} \hbar^{m-1} R_m^{(\pm)}(x, t)$ be two formal solutions of the Riccati equation associated with (2.7):

$$\begin{aligned} \hbar^2 \left(R^2 + \frac{\partial R}{\partial x} \right) &= \hbar f R + g \\ &= \hbar \left(\frac{\hbar R - (q^2 + p + t/2)}{x - q} + (x - q) \right) + x^4 + t x^2 - 2\theta x + 2H + \frac{t^2}{4}. \end{aligned} \tag{2.13}$$

The first two terms of $R^{(\pm)}$ are given explicitly as

$$R_0^{(\pm)}(x,t) = \pm(x-q_0)\sqrt{r(x,t)}, \quad R_1^{(\pm)}(x,t) = \frac{-(x+q_0) \pm \sqrt{r(x,t)}}{2r(x,t)},$$

where

$$r(x,t) = x^2 + 2q_0 x + 3q_0^2 + t.$$

The coefficients $R_m^{(\pm)}(x,t)$ for $m \geq 2$ are recursively determined by

$$\sum_{\substack{a+b=m+1 \\ a,b\geq 0}} R_a^{(\pm)} R_b^{(\pm)} + \frac{\partial R_m^{(\pm)}}{\partial x} = \sum_{\substack{a+b=m+1 \\ a,b\geq 0}} f_a R_b^{(\pm)} + g_{m+1}. \tag{2.14}$$

It follows from the recursion relation (2.14) that the coefficients $\{R_m^{(\pm)}\}_{m\geq 0}$ are (multi-valued) holomorphic on the domain $\mathbb{C} \setminus \{q_0, v_1, v_2\}$ as functions of x. Here v_1, v_2 are defined as zeros of $r(x,t)$. However, they are in fact holomorphic at $x = q_0$ as a consequence of [23, Theorem 1.1] (c.f., Remark 2.3). Note also that the assumptions (2.5) and (2.6) imply that q_0, v_1 and v_2 are pair-wise distinct.

Lemma 2.4. *The coefficients $\{R_m^{(\pm)}\}_{m\geq 0}$ have the following asymptotic behavior when x tends to ∞:*

$$R_0^{(\pm)}(x,t) = \pm\left(x^2 + \frac{t}{2} - \frac{\theta}{x} + O(x^{-2})\right)$$
$$R_1^{(\pm)}(x,t) = O(x^{-1}), \quad R_m^{(\pm)}(x,t) = O(x^{-2}) \quad \textit{for } m \geq 2.$$

Lemma 2.4 can be verified by using the recursion relation (2.14).

Using Lemma 2.4, now we fix a normalization (i.e., choice of the lower end-point of the integral in (2.12)) so that the WKB solution satisfies both of (2.7) and (2.8). Define

$$\tilde{S}^{(\pm)}(x,t,\hbar) = \sum_{m=0}^{\infty} \hbar^{m-1} \tilde{S}_m^{(\pm)}(x,t), \tag{2.15}$$

where

$$\tilde{S}_0^{(\pm)}(x,t) = \pm\frac{x(x-q_0)+t}{3}\sqrt{r(x,t)} \mp \frac{\theta}{2}\log\left(\frac{x+q_0+\sqrt{r(x,t)}}{x+q_0-\sqrt{r(x,t)}}\right) + \frac{2q_0^3}{3} + \frac{\theta}{2}\log q_0, \tag{2.16}$$

$$\tilde{S}_1^{(\pm)}(x,t) = -\frac{1}{4}\log r(x,t) \pm \frac{1}{4}\log\left(\frac{x+q_0+\sqrt{r(x,t)}}{x+q_0-\sqrt{r(x,t)}}\right) - \frac{1}{4}\log q_0, \tag{2.17}$$

$$\tilde{S}_m^{(\pm)}(x,t) = \int_{\infty^{(\pm)}}^{x} R_m^{(\pm)}(x,t)dx \quad \text{for } m \geq 2. \tag{2.18}$$

(Here we put ˜ on these functions to distinguish them from functions defined by topological recursion in next section.) Since the functions $R_m^{(\pm)}$ are multi-valued functions of x, the path of integration in (2.18) should be considered as a path on the Riemann surface of $R_0^{(\pm)}$. The Riemann surface is explicitly defined by the algebraic equation

$$y^2 = (x - q_0)^2 r(x,t). \tag{2.19}$$

We call the Riemann surface (2.19) the *semi-classical limit*, or the *spectral curve* of (2.7). The symbol $\infty^{(\pm)}$ in (2.18) represents the points on the spectral curve corresponding to $x = \infty$ satisfying

$$(x - q_0)\sqrt{r(x,t)} = \pm x^2 \left(1 + O(x^{-1})\right), \quad \text{as } x \to \infty^{(\pm)}.$$

The following Theorem is a modification of a result shown in [15].

Theorem 2.5 (C.f., [15, Section 4.2]). *The formal series*

$$\psi_\pm(x,t,\hbar) = \exp\left(\tilde{S}^{(\pm)}(x,t,\hbar)\right) \tag{2.20}$$

gives WKB solutions of isomonodromy system (2.7) *and* (2.8). *That is, the formal power series* $\tilde{S}^{(\pm)}(x,t,\hbar)$ *satisfies*

$$\frac{\partial}{\partial x}\tilde{S}^{(\pm)}(x,t,\hbar) = R^{(\pm)}(x,t,\hbar), \tag{2.21}$$

$$\hbar\frac{\partial \tilde{S}^{(\pm)}}{\partial t} = \frac{1}{2(x-q)}\left(\hbar\frac{\partial \tilde{S}^{(\pm)}}{\partial x} - \left(p + q^2 + \frac{t}{2}\right)\right) - \frac{q}{2}. \tag{2.22}$$

Proof. The equality (2.21) follows immediately from the definition of $\tilde{S}$. The equality (2.22) is equivalent to

$$\frac{\partial \tilde{S}_m^{(\pm)}}{\partial t} = \left[\frac{1}{2(x-q)}\left(\hbar\frac{\partial \tilde{S}^{(\pm)}}{\partial x} - \left(p + q^2 + \frac{t}{2}\right)\right) - \frac{q}{2}\right]_{\hbar^m} \tag{2.23}$$

for all $m \geq 0$. Here, for a formal series $c(\hbar) = \sum_{k=0}^{\infty} \hbar^k c_k$, we define $[c(\hbar)]_{\hbar^m} = c_m$.

The equalities (2.23) for $m = 0, 1$ are easily verified after applying $\partial/\partial t$ to (2.16) and (2.17). To compute $\partial \tilde{S}_m^{(\pm)}/\partial t$ for $m \geq 2$, we use the following identity for $R^{(\pm)}$.

Lemma 2.6 ([23, Proposition 1.2]). *The formal series* $R^{(\pm)}$ *satisfies*

$$\hbar\frac{\partial R^{(\pm)}}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\hbar R^{(\pm)} - (p + q^2 + t/2)}{2(x-q)}\right). \tag{2.24}$$

Lemma 2.6 implies

$$
\begin{aligned}
\frac{\partial}{\partial t}\tilde{S}_m^{(\pm)}(x,t) &= \int_{\infty^{(\pm)}}^{x} \frac{\partial}{\partial t} R_m^{(\pm)}(x,t)dx \\
&= \left[\frac{\hbar R^{(\pm)} - (p+q^2+t/2)}{2(x-q)}\right]_{\hbar^m} - \lim_{x\to\infty^{(\pm)}} \left[\frac{\hbar R^{(\pm)} - (p+q^2+t/2)}{2(x-q)}\right]_{\hbar^m}.
\end{aligned}
$$

Since

$$
\left[\frac{1}{2(x-q)}\right]_{\hbar^m} = \frac{q_m}{2x^2} + O(x^{-3}), \quad R_0^{(\pm)} = x^2\left(1+O(x^{-1})\right)
$$

holds as $x \to \infty^{(\pm)}$, we obtain

$$
\lim_{x\to\infty^{(\pm)}} \left[\frac{\hbar R^{(\pm)} - \left(p+q^2+t/2\right)}{2(x-q)}\right]_{\hbar^m} = \frac{q_m}{2} \quad \text{for } m \geq 1.
$$

Thus we have proved (2.23) for $m \geq 2$. □

The equalities (2.21) and (2.22) show that the Riccati equation (2.13) is written as

$$
\hbar^2 \left(\left(\frac{\partial \tilde{S}}{\partial x}\right)^2 + \frac{\partial^2 \tilde{S}}{\partial x^2} \right) = \hbar \left(2\hbar \frac{\partial \tilde{S}}{\partial t} + x \right) + x^4 + tx^2 - 2\theta x + 2H + \frac{t^2}{4}. \tag{2.25}
$$

§ 3. Quantum curve theorem

In this section we apply the Eynard-Orantin's topological recursion [11] for the spectral curve (2.19), and formulate our main theorem on the reconstruction of the whole isomonodromy system as a quantum curve.

§ 3.1. Topological recursion

The *topological recursion* is an algorithm associating some differential forms $W_{g,n}$ and numbers F_g to the following given data:

- A plane curve $(\mathcal{C}, x, y)$: $\mathcal{C}$ is a compact Riemann surface, $x, y : \mathcal{C} \to \mathbb{P}^1$ are meromorphic functions.
- The Bergman kernel B: It is a symmetric differential form on $\mathcal{C} \times \mathcal{C}$ with poles of order 2 along the diagonal, and satisfying some normalization conditions.

For our purpose, we choose $\mathcal{C} = \mathbb{P}^1$ and x, y are rational functions which parametrize the spectral curve (2.19) (called *Jimbo-Miwa curve* in [17]):

$$
\begin{cases} x(z) = -q_0 + \alpha(z + z^{-1}) \\ y(z) = \alpha(z - z^{-1})\left(-2q_0 + \alpha(z+z^{-1})\right), \end{cases} \tag{3.1}
$$

where z is a coordinate on $\mathbb{P}^1$, and

$$\alpha = \frac{1}{2}\left(-\frac{\theta}{q_0}\right)^{\frac{1}{2}}.$$

Note that $\alpha \neq 0$ under the assumption (2.5). The Bergman kernel is given by

$$B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$$

since $\mathcal{C}$ is of genus 0. Zeros of dx are called *ramification points* of the spectral curve (3.1). Our spectral curve has ramification points at $z = \pm 1$.

The topological recursion for our spectral curve (3.1) is formulated as follows (see [11] for general case):

Definition 3.1 ([11, Definition 4.2]). The *Eynard-Orantin differential* $W_{g,n}(z_1, \ldots, z_n)$ of type (g, n) is a meromorphic n-differential on the n-times product of the spectral curve (3.1) defined by the following *topological recursion relation*:

- For $2g - 2 + n \leq 0$:

$$W_{0,1}(z_1) = y(z_1)dx(z_1).$$
$$W_{0,2}(z_1, z_2) = B(z_1, z_2).$$

- For $2g - 2 + n = 1$:

$$W_{0,3}(z_1, z_2, z_3) = \frac{1}{2\pi i} \sum_{a \in \{\pm 1\}} \oint_{\gamma_a} K(z, z_1) \times \Big[W_{0,2}(z, z_2) W_{0,2}(\bar{z}, z_3) + W_{0,2}(z, z_3) W_{0,2}(\bar{z}, z_2) \Big].$$

$$W_{1,1}(z_1) = \frac{1}{2\pi i} \sum_{a \in \{\pm 1\}} \oint_{\gamma_a} K(z, z_1) W_{0,2}(z, \bar{z}).$$

- For $2g - 2 + n \geq 2$:

$$W_{g,n}(z_1, \ldots, z_n) = \frac{1}{2\pi i} \sum_{a \in \{\pm 1\}} \oint_{\gamma_a} K(z, z_1) \times \Bigg[\sum_{j=2}^{n} \Big(W_{0,2}(z, z_j) W_{g,n-1}(\bar{z}, z_{[\hat{1},\hat{j}]}) + W_{0,2}(\bar{z}, z_j) W_{g,n-1}(z, z_{[\hat{1},\hat{j}]}) \Big) + W_{g-1,n+1}(z, \bar{z}, z_{[\hat{1}]}) + \sum_{\substack{g_1 + g_2 = g \\ I \sqcup J = [\hat{1}]}}^{\text{stable}} W_{g_1,|I|+1}(z, z_I) W_{g_2,|J|+1}(\bar{z}, z_J) \Bigg]. \tag{3.2}$$

Here $\gamma_{\pm 1}$ is a small cycle on z-plane which encircles the ramification point $z = \pm 1$ in the counter-clockwise direction, $\bar{z} = 1/z$ is the conjugate of z near the ramification point, and the *recursion kernel* $K(z, z_1)$ is given by

$$K(z, z_1) = \frac{1}{2(y(z) - y(\bar{z}))dx(z)} \int_{w=\bar{z}}^{z} W_{0,2}(w, z_1).$$

Also, we use the index convention $[\hat{j}] = \{1, \ldots, n\} \setminus \{j\}$ and so on. Lastly, the sum in the third line of (3.2) is taken for indices in the stable range (i.e., only $W_{g,n}$'s with $2g - 2 + n \geq 1$ appear).

Explicit expressions of $W_{g,n}$ for $2g - 2 + n = 1$ are given by

$$\begin{aligned} W_{0,3}(z_1, z_2, z_3) = \Bigg(& \frac{1}{(q_0 - \alpha)(z_1 - 1)^2(z_2 - 1)^2(z_3 - 1)^2} \\ & - \frac{1}{(q_0 + \alpha)(z_1 + 1)^2(z_2 + 1)^2(z_3 + 1)^2} \Bigg) \frac{dz_1 dz_2 dz_3}{8\alpha^2}, \end{aligned}$$

$$\begin{aligned} W_{1,1}(z_1) = \Big(& \alpha^3 z_1^6 + 2\alpha^2 q_0 z_1^5 + (4\alpha q_0^2 - 5\alpha^3) z_1^4 + (8q_0^3 - 12\alpha^2 q_0) z_1^3 \\ & + (4\alpha q_0^2 - 5\alpha^3) z_1^2 + 2\alpha^2 q_0 z_1 + \alpha^3 \Big) \frac{dz_1}{32\alpha^2 (q_0^2 - \alpha^2)(z_1^2 - 1)^4}. \end{aligned}$$

Eynard-Orantin differentials have the following properties (see [11]):

- $W_{g,n}$ is symmetric under permutations of variables $z_1, \ldots, z_n$.
- For $2g - 2 + n \geq 1$, $W_{g,n}$ is anti-invariant under the involution $z_j \mapsto \bar{z}_j$ for each variable:

$$W_{g,n}(z_1, \ldots, \bar{z}_j, \ldots, z_n) = -W_{g,n}(z_1, \ldots, z_j, \ldots, z_n) \ \text{ for } j = 1, \ldots, n. \tag{3.3}$$

- As a differential form on each variable z_j, $W_{g,n}$, for $2g - 2 + n \geq 1$, is holomorphic except for the ramification point $z_j = \pm 1$, and may have a pole at those points. In particular, we have

$$W_{g,n}(z_1, \ldots, z_n) = \frac{c_{g,n}}{z_j^2} \left(1 + O(z_j^{-1})\right) dz_1 \cdots dz_n \tag{3.4}$$

  as $z_j \to \infty$ for each $j = 1, \ldots, n$.

- $W_{g,n}$ is also holomorphic in t and θ on the domain specified by the conditions (2.5) and (2.6). Formulas describing derivatives of $W_{g,n}$ with respect to t and θ will be given in Section 3.4.

§ 3.2. Topological recursion and quantum curve theorem

For $g \geq 0, n \geq 1$ satisfying $2g-2+n \geq 1$, define

$$F_{g,n}(z_1,\ldots,z_n) = \int_{\infty}^{z_1} \cdots \int_{\infty}^{z_n} W_{g,n}(z_1,\ldots,z_n). \tag{3.5}$$

Obviously, $F_{g,n}$ satisfies

$$d_{z_1} \cdots d_{z_n} F_{g,n}(z_1,\ldots,z_n) = W_{g,n}(z_1,\ldots,z_n). \tag{3.6}$$

It also follows from (3.4) that

$$F_{g,n}(z_1,\ldots,z_n) = \frac{c'_{g,n}}{z_j}\left(1 + O(z_j^{-1})\right) \tag{3.7}$$

holds when $z_j \to \infty$ for each $j = 1,\ldots,n$.

To compare $F_{g,n}$ and $\tilde{S}$ defined in (2.15), we also fix the branch of $\sqrt{r(x,t)}$ so that $R_0^{(+)}(x,t) = y(z(x))$ holds. Here $z(x)$ be the inverse function of $x(z)$ given in (3.1) whose branch is chosen so that $z(x) \to \infty$ holds as $x \to \infty$. We will omit the superscript $(+)$ and denote $\tilde{S}(x,t,\hbar) = \tilde{S}^{(+)}(x,t,\hbar)$ for simplicity. Then, our main result is formulated as follows:

Theorem 3.2. *Define*

$$S(x,t,\hbar) = \sum_{m=0}^{\infty} \hbar^{m-1} S_m(x,t). \tag{3.8}$$

by

$$S_0(x,t) = \tilde{S}_0(x,t), \tag{3.9}$$

$$S_1(x,t) = \tilde{S}_1(x,t), \tag{3.10}$$

$$S_m(x,t) = \sum_{\substack{g \geq 0, n \geq 1, \\ 2g-2+n=m-1}} \frac{F_{g,n}(z(x),\ldots,z(x))}{n!} \quad \text{for } m \geq 2. \tag{3.11}$$

Here $\tilde{S}_0$ and $\tilde{S}_1$ are given in (2.16) *and* (2.17)*. Then, the formal series*

$$\psi(x,t,\hbar) = \exp\left(S(x,t,\hbar)\right) \tag{3.12}$$

coincides with the WKB solution ψ_+ (defined by (2.20)*) of the isomonodromy system* (2.7) *and* (2.8) *with the 0-parameter solution* (2.3) *being substituted. In particular, we have*

$$S(x,t,\hbar) = \tilde{S}(x,t,\hbar). \tag{3.13}$$

Therefore, the generating function of $F_{g,n}$'s gives the WKB solution of the isomonodromy system associated with P_2, whose semi-classical limit is the spectral curve (2.19) for the initial data of the topological recursion. Therefore we have seen that the isomonodromy system is a *quantum curve* of (2.19) in the sense of [6, 7].

Remark 3.3. In the case of the first Painlevé equation discussed in [19] the function $F_{g,n}$ was defined by the formula (5.1) below instead of (3.5). In Section 5 we will discuss how the quantum curve changes if we change the definition of $F_{g,n}$.

Remark 3.4. Once we have established the relation between ψ_+ constructed in (2.20) and that constructed from topological recursion (c.f., (3.13)), we can also show that the other WKB solution ψ_- of the isomonodromy system is also obtained from the topological recursion. This is because ψ_+ and ψ_- are related by term-wise analytic continuation from x to $\check{x}$ on the spectral curve. Here we identify x and $\check{x}$ with points on the spectral curve satisfying $\check{x} = \iota(x)$, where ι is the covering involution of the spectral curve (i.e., $R_0^{(+)}(\check{x}, t) = R^{(-)}(x, t)$). On the topological recursion side, the covering involution is realized by the conjugation $z(x) \mapsto \bar{z}(x) = 1/z(x)$. Consequently, we have

$$\frac{1}{\hbar}\tilde{S}_0^{(-)}(x,t) + \tilde{S}_1^{(-)}(x,t) + \sum_{\substack{g\geq 0, n\geq 1\\ 2g-2+n\geq 1}} \hbar^{2g-2+n}\frac{F_{g,n}(\bar{z}(x), \ldots, \bar{z}(x))}{n!} = \tilde{S}^{(-)}(x,t,\hbar).$$

Explicit computation shows

$$\begin{aligned}(3.14)\qquad & \frac{F_{0,3}(z,z,z)}{3!} + F_{1,1}(z) \\ &= \frac{-\alpha^3 z(3z^4 - 14z^2 - 9) - \alpha^2 q_0(3z^4 - 24z^2 + 1) - 4\alpha q_0^2 z(2z^2+3) - 2q_0^3(9z^2+1)}{96\alpha^2(z^2-1)^3(q_0^2-\alpha^2)^2},\end{aligned}$$

and we can check that (3.14) coincides with $\tilde{S}_2(x,t)$ after the substitution $z \mapsto z(x)$. A proof of Theorem 3.2 in full order will be given in Section 4. The rest of this section is devoted to describe some properties of $W_{g,n}$ and $F_{g,n}$ which will be used in the proof.

§ 3.3. Free energies and the τ-function

Let us recall the notion of free energies of the spectral curve (3.1).

Definition 3.5 ([11, Definition 4.3]). Define the *(closed) free energy* $F_g = F_g(t)$ for $g \geq 2$ by

$$F_g(t) = \frac{1}{2\pi i(2-2g)} \sum_{a\in\{\pm 1\}} \oint_{\gamma_a} \Phi(z) W_{g,1}(z), \tag{3.15}$$

where

$$\Phi(z) = \int_{z_o}^{z} y(z)dx(z)$$

and z_o is a generic point. Free energies F_g for $g = 0, 1$ are also defined but in a different manner (see [11, §4.2.2 and §4.2.3] for the definition).

Note that F_g is different from $F_{g,n}$ defined in the previous subsection. F_g's are also called *symplectic invariants* since they are invariant under symplectic transformations of the spectral curve (see [11]).

The following theorem will be used in the proof of our main theorem.

Theorem 3.6 ([17, Theorem 3.4]; see also [18]). *The generating function of the free energies*

$$\log \tau(t, \hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g(t)$$

gives the logarithm of the τ-function of P_2. That is, we have

$$\frac{dF_g(t)}{dt} = \sigma_{2g}(t) \quad \text{for } g \geq 0. \tag{3.16}$$

Here $\sigma_{2g}(t)$ is defined in (2.11).

We note that the relation between free energies and τ-functions was first shown in [3] for a special case of P_2 with $\theta = 0$. The result is generalized to P_2 with non-zero θ in [17], and to all second order Painlevé equations in [18]. It is worth mentioning that, although the τ-function is defined up to constant (see (2.10)), the topological recursion fixes the constant and specifies one particular τ-function (see [17, Theorem 3.4 and 4.2]).

§ 3.4. Variational formulas

Since the spectral curve (3.1) depends on the parameter t, so does $W_{g,n}$. The following formula (shown in [11]; see also [4]) describes the derivation of $W_{g,n}$ with respect to t.

Proposition 3.7 (C.f., [11, Theorem 5.1]). *Set*

$$\Lambda(z) = -\frac{\alpha}{2}(z - z^{-1}). \tag{3.17}$$

Then, for $2g - 2 + n \geq 0$, we have

$$\begin{aligned} &\frac{\partial}{\partial t} W_{g,n}(z(x_1), \ldots, z(x_n)) \\ &\qquad = \left(\frac{1}{2\pi i} \int_{\Gamma} \Lambda(w) W_{g,n+1}(z_1, \ldots, z_n, w) \right) \Bigg|_{(z_1,\ldots,z_n)=(z(x_1),\ldots,z(x_n))}. \end{aligned} \tag{3.18}$$

$$\frac{d}{dt} F_g = \frac{1}{2\pi i} \int_{\Gamma} \Lambda(w) W_{g,1}(w). \tag{3.19}$$

Here Γ *is a cycle on* w*-plane which is given by the sum of two positively oriented small circles around the origin and infinity. That is,*

$$\frac{1}{2\pi i}\int_{\Gamma} f(w)dw = \operatorname*{Res}_{w=\infty} f(w)dw + \operatorname*{Res}_{w=0} f(w)dw.$$

Proof. We can check that $\Lambda(z)$ satisfies the required condition

$$\frac{\partial x}{\partial t}(z)dy(z) - \frac{\partial y}{\partial t}(z)dx(z) = \int_{\Gamma} \Lambda(w) W_{0,2}(z,w)$$

to apply [11, Theorem 5.1]. Then the equalities (3.18) and (3.19) immediately follows from the result. □

Integrating (3.18) with respect to $x_1, \ldots, x_n$ and using (3.3), we have

$$\begin{aligned} \frac{\partial}{\partial t} F_{g,n}(z(x_1), \ldots, z(x_n)) &= \frac{1}{2\pi i}\int_{\Gamma} \Lambda(w) \frac{\partial}{\partial w} F_{g,n+1}(z(x_1), \ldots, z(x_n), w) dw \\ &= \lim_{w\to\infty} \big(-2w\Lambda(w) \frac{\partial}{\partial w} F_{g,n+1}(z(x_1), \ldots, z(x_n), w)\big). \end{aligned} \tag{3.20}$$

$$\frac{d}{dt} F_g = \lim_{w\to\infty} \big(-2w\Lambda(w) \frac{\partial}{\partial w} F_{g,1}(w)\big). \tag{3.21}$$

The variation formula for the other parameter θ is also described as follows.

Proposition 3.8 (C.f., [11, Theorem 5.1])**.** *For* $2g-2+n \geq 0$*, we have*

$$\begin{aligned} &\frac{\partial}{\partial \theta} W_{g,n}(z(x_1), \ldots, z(x_n)) \\ &\qquad = \left(\int_0^{\infty} W_{g,n+1}(z_1, \ldots, z_n, w)\right)\bigg|_{(z_1,\ldots,z_n)=(z(x_1),\ldots,z(x_n))}. \end{aligned} \tag{3.22}$$

$$\frac{\partial}{\partial \theta} F_g = \int_0^{\infty} W_{g,1}(w). \tag{3.23}$$

The above follows from the fact that the parameter θ satisfies

$$\theta = \frac{1}{2\pi i}\oint_{\mathcal{A}} R_0^{(+)}(x)dx,$$

where $\mathcal{A}$ is a closed cycle on the spectral curve encircling the two branch points (c.f., [11, Section 5.3]). Such a parameter is called a *filling fraction*. Proposition 3.8 will be used to prove another version of quantum curve theorem in Section 5.

§ 4. Proof of the main theorem

In this section we will give a proof of Theorem 3.2. The proof is rather technical, and similar to the one given in [19] for the case of the first Painlevé equation. However, it is worth describing our proof concretely since the definition (3.5) of $F_{g,n}$ discussed in this article is different from the one used in [19]. (As is mentioned in Introduction, the resulting quantum curve changes completely if we change the normalization of $F_{g,n}$.)

§ 4.1. Differential recursion for $F_{g,n}$

Topological recursion (3.2) implies that the function $F_{g,n}(z_1, \ldots, z_n)$ defined in (3.5) satisfies the following *differential recursion relation* (c.f., [6, 7, 19]).

Proposition 4.1 (C.f., [19, Theorem 3.11]). *For $2g-2+n \geq 2$ with $g \geq 0$, $n \geq 1$, we have*

(4.1)
$$\begin{aligned}
\frac{\partial F_{g,n}}{\partial z_1}(z_1, \ldots, z_n) = &-\sum_{j=2}^{n} \frac{1}{2y(z_1)\frac{dx}{dz}(z_1)} \left(\frac{1}{z_1 - z_j} + \frac{1}{z_1^2(\bar{z}_1 - z_j)} \right) \frac{\partial F_{g,n-1}}{\partial z_1}(z_1, z_{[\hat{1},\hat{j}]}) \\
&+ \sum_{j=2}^{n} \frac{1}{2y(z_j)\frac{dx}{dz}(z_j)} \left(\frac{1}{z_1 - z_j} - \frac{1}{z_1 - \bar{z}_j} \right) \frac{\partial F_{g,n-1}}{\partial z_1}(z_j, z_{[\hat{1},\hat{j}]}) \\
&- \frac{1}{2y(z_1)\frac{dx}{dz}(z_1)} \frac{\partial^2}{\partial u_1 \partial u_2} \left(F_{g-1,n+1}(u_1, u_2, z_{[\hat{1}]}) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = [\hat{1}]}}^{\text{stable}} F_{g_1,|I|+1}(u_1, z_I) F_{g_2,|J|+1}(u_2, z_J) \right) \Bigg|_{u_1=u_2=z_1} \\
&+ \frac{1}{2\frac{dy}{dz}(s)\frac{dx}{dz}(s)} \left(\frac{1}{z_1 - s} - \frac{1}{z_1 - \bar{s}} \right) \Bigg[\sum_{j=2}^{n} \left(\frac{1}{s - z_j} + \frac{1}{s^2(\bar{s} - z_j)} \right) \frac{\partial F_{g,n-1}}{\partial z_1}(s, z_{[\hat{1},\hat{j}]}) \\
&+ \frac{\partial^2}{\partial u_1 \partial u_2} \left(F_{g-1,n+1}(u_1, u_2, z_{[\hat{1}]}) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = [\hat{1}]}}^{\text{stable}} F_{g_1,|I|+1}(u_1, z_I) F_{g_2,|J|+1}(u_2, z_J) \right) \Bigg|_{u_1=u_2=s} \Bigg].
\end{aligned}$$

Here s is defined as one of solutions of $x(z) = q_0$. That is,

$$x(z) - q_0 = \frac{\alpha}{z}(z - s)(z - \bar{s}).$$

We can show the above equality similarly to the proof of [19, Theorem 3.11] (see also [6, 7]).

§ 4.2. Variation formula for $F_{g,n}$

Proposition 4.2 (C.f., [19, Theorem 3.13]). *For $2g-2+n \geq 1$ with $g \geq 0$, $n \geq 1$, we have*

$$\frac{\partial}{\partial t} F_{g,n}(z(x_1), \ldots, z(x_n)) = E_{g,n}(z(x_1), \ldots, z(x_n)), \tag{4.2}$$

where

$$(4.3)\quad E_{g,n}(z_1,\dots,z_n)=$$
$$\sum_{j=1}^{n}\frac{\alpha(z_j-\bar z_j)}{2y(z_j)\frac{dx}{dz}(z_j)}\frac{\partial F_{g,n}}{\partial z_j}(z_1,\dots,z_n)+\frac{\alpha(s-\bar s)}{2\frac{dy}{dz}(s)\frac{dx}{dz}(s)}\left[\sum_{j=1}^{n}\left(\frac{1}{s-z_j}+\frac{1}{s^2(\bar s-z_j)}\right)\frac{\partial F_{g,n}}{\partial z_1}(s,z_{[\hat j]})\right.$$
$$\left.+\frac{\partial^2}{\partial u_1\partial u_2}\left(F_{g-1,n+2}(u_1,u_2,z_1,\dots,z_n)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{1,\dots,n\}}}^{\text{stable}}F_{g_1,|I|+1}(u_1,z_I)F_{g_2,|J|+1}(u_2,z_J)\right)\Bigg|_{u_1=u_2=s}\right].$$

Proof. It follows from the definition (3.17) of $\Lambda(z)$ that

$$\frac{w\Lambda(w)}{y(w)\frac{dx}{dz}(w)}=O(1),\quad w\Lambda(w)\left(\frac{1}{w-u}-\frac{1}{w-v}\right)=-\frac{\alpha}{2}(u-v)+O(w^{-1})$$

hold when w tends to infinity. Then, the desired equality follows from the asymptotic behavior (3.4) of $W_{g,n}$, the variation formula (3.20), and the differential recursion relation (4.1) with n being replaced by $n+1$. □

As a corollary of (4.2), we obtain

$$(4.4)\qquad \sum_{\substack{2g-2+n=m-1\\ g\geq 0,\ n\geq 1}}\frac{E_{g,n}(z(x),\dots,z(x))}{n!}=\frac{\partial S_m}{\partial t}(x,t)\quad\text{for } m\geq 2.$$

§ 4.3. Partial differential equation for S

Using Proposition 4.1 and 4.2, in this subsection we will show

Proposition 4.3. *The formal series $S(x,t,\hbar)$ defined in (3.8) satisfies*

$$(4.5)\qquad \hbar^2\left(\left(\frac{\partial S}{\partial x}\right)^2+\frac{\partial^2 S}{\partial x^2}\right)=\hbar\left(2\hbar\frac{\partial S}{\partial t}+x\right)+x^4+tx^2-2\theta x+2H+\frac{t^2}{4}.$$

The equality (4.5) is same as (2.25) which is satisfied by $\tilde S(x,t,\hbar)$ defined in (2.15). The is equivalent to the following hierarchy of equalities:

$$(4.6)\qquad \left(\frac{\partial S_0}{\partial x}\right)^2=x^4+tx^2-2\theta x+2[\sigma]_{\hbar^0}+\frac{t^2}{4}=(x-q_0)^2r(x,t),$$

$$(4.7)\qquad 2\frac{\partial S_0}{\partial x}\frac{\partial S_1}{\partial x}+\frac{\partial S_0^2}{\partial x^2}=2\frac{\partial S_0}{\partial t}+x+2[\sigma]_{\hbar^1},$$

$$(4.8)\qquad \sum_{\substack{a+b=m+1\\ a,b\geq 0}}\frac{\partial S_a}{\partial x}\frac{\partial S_b}{\partial x}+\frac{\partial^2 S_m}{\partial x^2}=2\frac{\partial S_m}{\partial t}+2[\sigma]_{\hbar^{m+1}}\quad\text{for } m\geq 1.$$

Recall that $[\bullet]_{\hbar^m}$ denotes the coefficient of $\hbar^m$ in a given formal series $\bullet$, and σ is defined in (2.9). As we have mentioned in Section 3.2, the equalities (4.6), (4.7) and (4.8) for $m = 1$ hold. To prove (4.8) for $m \geq 2$, we need to prepare several statements.

For $g \geq 0$, $n \geq 1$ satisfying $2g - 2 + n \geq 2$, we set

$$
\begin{aligned}
(4.9)\quad & G_{g,n}(z_1, \ldots, z_n) = \\
& \frac{\partial F_{g,n}}{\partial z_1}(z_1, \ldots, z_n) + \sum_{j=2}^{n} \frac{1}{2y(z_1)\frac{dx}{dz}(z_1)} \left(\frac{1}{z_1 - z_j} + \frac{1}{z_1^2(\bar{z}_1 - z_j)} \right) \frac{\partial F_{g,n-1}}{\partial z_1}(z_1, z_{[\hat{1},\hat{j}]}) \\
& \quad - \sum_{j=2}^{n} \frac{1}{2y(z_j)\frac{dx}{dz}(z_j)} \left(\frac{1}{z_1 - z_j} - \frac{1}{z_1 - \bar{z}_j} \right) \frac{\partial F_{g,n-1}}{\partial z_1}(z_j, z_{[\hat{1},\hat{j}]}) \\
& \quad + \frac{1}{2y(z_1)\frac{dx}{dz}(z_1)} \frac{\partial^2}{\partial u_1 \partial u_2} \Bigg(F_{g-1,n+1}(u_1, u_2, z_{[\hat{1}]}) \\
& \qquad + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = [\hat{1}]}}^{\text{stable}} F_{g_1,|I|+1}(u_1, z_I) F_{g_2,|J|+1}(u_2, z_J) \Bigg) \Bigg|_{u_1=u_2=z_1}.
\end{aligned}
$$

Lemma 4.4 (C.f., [19, Lemma 4.2])**.** *For $m \geq 2$ we have*

$$
(4.10)\quad \left(\frac{2y(z)}{\frac{dx}{dz}(z)} \sum_{\substack{2g-2+n=m \\ g\geq 0,\ n\geq 1}} \frac{G_{g,n}(z, \ldots, z)}{(n-1)!} \right) \Bigg|_{z=z(x)} = \sum_{\substack{a+b=m+1 \\ a,b\geq 0}} \frac{\partial S_a}{\partial x} \frac{\partial S_b}{\partial x} + \frac{\partial^2 S_m}{\partial x^2} - \frac{1}{x - q_0} \frac{\partial S_m}{\partial x}.
$$

Proof. The proof is similar to that of [19, Lemma 4.2]. Applying $\sum_{2g-2+m=m} \frac{1}{(n-1)!}$ and the principal specialization $(z_1, \ldots, z_n) \mapsto (z, \ldots, z)$ to (4.9), we obtain

$$
\begin{aligned}
& \left(\frac{2y(z)}{\frac{dx}{dz}(z)} \sum_{\substack{2g-2+n=m \\ g\geq 0,\ n\geq 1}} \frac{G_{g,n}(z, \ldots, z)}{(n-1)!} \right) \Bigg|_{z=z(x)} = 2y(z(x)) \frac{\partial S_{m+1}}{\partial x}(x) \\
& + \frac{1}{\frac{dx}{dz}(z(x))} \left(\frac{1}{z(x)} - \frac{\frac{dy}{dz}(z(x))}{y\big(z(x)\big)} \right) \frac{\partial S_m}{\partial x}(x) + \frac{\partial^2 S_m}{\partial x^2}(x) + \sum_{\substack{a+b=m+1 \\ a,b\geq 2}} \frac{\partial S_a}{\partial x}(x) \frac{\partial S_b}{\partial x}(x).
\end{aligned}
$$

(C.f., [6, Theorem 6.5].) Using the equalities

$$
\frac{\partial S_0}{\partial x}(x) = y(z(x)), \quad \frac{\partial S_1}{\partial x}(x) = -\frac{x + q_0}{2r(x)} + \frac{1}{2\sqrt{r(x)}} = -\frac{1}{\alpha z(z - z^{-1})^2} \Bigg|_{z=z(x)},
$$

we can verify

$$
\frac{1}{\frac{dx}{dz}(z(x))} \left(\frac{1}{z(x)} - \frac{\frac{dy}{dz}(z(x))}{y\big(z(x)\big)} \right) = -\frac{1}{x - q_0} + 2 \frac{\partial S_1}{\partial x}(x).
$$

Thus we have proved (4.10). □

Lemma 4.5 (C.f., [19, Lemma 4.3]). *For $m \geq 2$, we have*

$$\sum_{\substack{2g-2+n=m\\ g\geq 0,\ n\geq 2}} \left(\frac{2y(z)}{\frac{dx}{dz}(z)} \frac{G_{g,n}(z,\ldots,z)}{(n-1)!} - \frac{2E_{g,n-1}(z,\ldots,z)}{(n-1)!} \right)\Bigg|_{z=z(x)} = -\frac{1}{x-q_0}\frac{\partial S_m}{\partial x}. \tag{4.11}$$

Proof. First, we note that Proposition 4.1 implies

$$\begin{aligned} G_{g,n}(z_1,\ldots,z_n) = \\ & \frac{1}{2\frac{dy}{dz}(s)\frac{dx}{dz}(s)}\left(\frac{1}{z_1-s}-\frac{1}{z_1-\bar{s}}\right)\left[\sum_{j=2}^{n}\left(\frac{1}{s-z_j}+\frac{1}{s^2(\bar{s}-z_j)}\right)\frac{\partial F_{g,n-1}}{\partial z_1}(s,z_{[\hat{1},\hat{j}]})\right. \\ & \left. +\frac{\partial^2}{\partial u_1\partial u_2}\left(F_{g-1,n+1}(u_1,u_2,z_{[\hat{1}]})+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=[\hat{1}]}}^{\text{stable}} F_{g_1,|I|+1}(u_1,z_I)F_{g_2,|J|+1}(u_2,z_J)\right)\Bigg|_{u_1=u_2=s}\right]. \end{aligned} \tag{4.12}$$

Using the above expression of $G_{g,n}$, we can show that the equality

$$\begin{aligned} \frac{2y(z_1)}{\frac{dx}{dz}(z_1)}\frac{G_{g,n}(z_1,\ldots,z_n)}{(n-1)!} - \frac{2E_{g,n-1}(z_2,\ldots,z_n)}{(n-1)!} = \\ -\frac{1}{(n-1)!}\sum_{j=2}^{n}\frac{\alpha(s-\bar{s})}{y(z_j)\frac{dx}{dz}(z_j)}\frac{\partial F_{g,n-1}}{\partial z_j}(z_2,\ldots,z_n) \end{aligned} \tag{4.13}$$

holds for $g \geq 0$ and $n \geq 2$ satisfying $2g-2+n \geq 2$. Here we have used

$$\frac{y(z_1)}{\frac{dx}{dz}(z_1)}\left(\frac{1}{z_1-s}-\frac{1}{z_1-\bar{s}}\right) = \alpha(s-\bar{s}).$$

Note also that

$$\frac{\partial F_{g,n}}{\partial z_j}(z,\cdots,z) = \frac{1}{n}\frac{\partial}{\partial z}F_{g,n}(z,\ldots,z)$$

holds since $F_{g,n}(z_1,\ldots,z_n)$ is a symmetric function of $z_1,\ldots,z_n$ (c.f., [6, Lemma 6.4]). Using the equality, we obtain the desired relation (4.11) after setting $(z_1,\ldots,z_n) \mapsto (z,\ldots,z)$ and summing up (4.13) for $2g-2+n=m$ with $g \geq 0$, $n \geq 2$. □

Lemma 4.4, 4.5 and (4.4) imply

$$\sum_{\substack{a+b=m+1\\ a,b\geq 0}}\frac{\partial S_a}{\partial x}\frac{\partial S_b}{\partial x}+\frac{\partial^2 S_m}{\partial x^2}-2\frac{\partial S_m}{\partial t} = \begin{cases} 0 & \text{if } m \text{ is even.} \\ \frac{2y(z)}{\frac{dx}{dz}(z)}G_{(m+1)/2,1}(z) & \text{if } m \text{ is odd.} \end{cases} \tag{4.14}$$

Lemma 4.6 (C.f., [19, Lemma 4.5]). *For $g \geq 2$ we have*

$$\frac{2y(z)}{\frac{dx}{dz}(z)} G_{g,1}(z) = 2\sigma_{2g}(t).$$

Proof. By the second expression (4.12) of $G_{g,n}$, we have

$$\frac{2y(z)}{\frac{dx}{dz}(z)} G_{g,1}(z) = \frac{\alpha(s-\bar{s})}{\frac{dy}{dz}(s)\frac{dx}{dz}(s)} \left(\frac{\partial^2 F_{g-1,2}}{\partial z_1 \partial z_2}(s,s) + \sum_{\substack{g_1+g_2=g \\ g_1,g_2 \geq 1}} \frac{\partial F_{g_1,1}}{\partial z_1}(s) \frac{\partial F_{g_2,1}}{\partial z_1}(s) \right).$$

We can check that this coincides with $2(dF_g/dt)$ by using (3.21) and (4.1). Hence the desired equality follows from Theorem 3.6. □

We have already seen in (2.11) that $[\sigma]_{\hbar^{m+1}} = 0$ for even m. Therefore, (4.14) together with Lemma 4.6 imply (4.8) for $m \geq 2$.

§ 4.4. Proof of quantum curve theorem

Here we show that Proposition 4.3 implies Theorem 3.2. That is, we will prove

Proposition 4.7. *Let $S(x,t,\hbar)$ (resp., $\tilde{S}(x,t,\hbar) = \tilde{S}^{(+)}(x,t,\hbar)$) be the formal series defined in* (3.8) *(resp.,* (2.15)*). Then we have*

$$S(x,t,\hbar) = \tilde{S}(x,t,\hbar). \tag{4.15}$$

Proof. The equality (4.15) is equivalent to

$$S_m(x,t) = \tilde{S}_m(x,t) \tag{4.16}$$

for any $m \geq 0$. We will prove (4.16) by induction.

As is mentioned in Section 3.2, we have already checked that (4.16) holds for $m = 0, 1, 2$. Let $k \geq 2$ an integer, and suppose that (4.16) holds for $m = 0, \ldots, k$. In particular, the assumption for $m = k$ implies

$$S_k(x,t) = \int_\infty^x R_k(x,t)dx.$$

Then, as is proved in (2.23), we have

$$\frac{\partial S_k}{\partial t} = \left[\frac{1}{2(x-q)} \left(\hbar \frac{\partial S}{\partial x} - \left(p + q^2 + \frac{t}{2} \right) \right) - \frac{q}{2} \right]_{\hbar^k}.$$

Therefore, the equality (4.8) for $m = k$ implies that, under the induction hypothesis, $\partial S_{k+1}/\partial x$ satisfies the same equation (2.14) satisfied by R_{k+1}. Then the uniqueness of the solution shows

$$\frac{\partial S_{k+1}}{\partial x}(x,t) = R_{k+1}(x,t).$$

Since we know that $\lim_{x\to\infty} S_{k+1}(x,t) = 0$ from its definition (see (3.7)), we can conclude that (4.16) also holds for $m = k+1$. This completes the proof of Proposition 4.7, and thus we have proved Theorem 3.2. $\square$

§ 5. Another quantum curve theorem

Let $W_{g,n}$ be the Eynard-Orantin differentials for the spectral curve (3.1) defined in Section 3.1. Here we introduce functions

$$\bar{F}_{g,n}(z_1,\ldots,z_n) = \frac{1}{2^n}\int_{\bar{z}_1}^{z_1}\cdots\int_{\bar{z}_n}^{z_n} W_{g,n}(z_1,\ldots,z_n) \tag{5.1}$$

for $g \geq 0$, $n \geq 1$ with $2g-2+n \geq 1$. Thanks to the property (3.3) of $W_{g,n}$, the function $\bar{F}_{g,n}$ satisfies the following equality as well as $F_{g,n}$ (c.f., (3.6)):

$$d_{z_1}\cdots d_{z_n}\bar{F}_{g,n}(z_1,\ldots,z_n) = W_{g,n}(z_1,\ldots,z_n). \tag{5.2}$$

In the case of the first Painlevé equation discussed in [19] the function (5.1) was used to obtain a quantum curve instead of (3.5). $F_{g,n}$ and $\bar{F}_{g,n}$ coincide in the case of the first Painlevé equation since the spectral curve has only one ramification point (see [19, Remark 4.8]). However, in the case of the second Painlevé equation, it turns out that $\bar{F}_{g,n}$'s also give a quantum curve which is different from the one given in Theorem 3.2.

We put $^-$ on quantities appearing here to distinguish from those appearing in previous sections. The quantum curve theorem corresponding to $\bar{F}_{g,n}$ is formulated as follows.

Theorem 5.1. *Define $\bar{S}(x,t,\hbar) = \sum_{m=0}^{\infty}\hbar^{m-1}\bar{S}_m(x,t)$ by*

$$\bar{S}_0(x,t) = \frac{x(x-q_0)+t}{3}\sqrt{r(x,t)} - \frac{\theta}{2}\log\left(\frac{x+q_0+\sqrt{r(x,t)}}{x+q_0-\sqrt{r(x,t)}}\right) + \frac{2q_0^3}{3} + \frac{\theta}{2}\log q_0, \tag{5.3}$$

$$\bar{S}_1(x,t) = -\frac{1}{4}\log r(x,t) - \frac{1}{4}\log q_0, \tag{5.4}$$

$$\bar{S}_m(x,t) = \sum_{\substack{g\geq 0, n\geq 1,\\ 2g-2+n=m-1}} \frac{\bar{F}_{g,n}(z(x),\ldots,z(x))}{n!} \quad \text{for } m \geq 2. \tag{5.5}$$

Here we fix the branch of $z(x)$ and $\sqrt{r(x,t)}$ similarly to Theorem 3.2. We also set

$$a(t,\hbar) = \sum_{m=2}^{\infty}\hbar^{m-1}\lim_{x\to\infty}\bar{S}_m(x,t).$$

Then, the formal series

$$\bar{\psi}(x,t,\hbar) = \exp(\bar{S}(x,t,\hbar) - a(t,\hbar)) \tag{5.6}$$

satisfies the following isomonodromy system

$$\hbar^2 \frac{\partial^2 \bar{\psi}}{\partial x^2} = \hbar \bar{f} \frac{\partial \bar{\psi}}{\partial x} + \bar{g} \bar{\psi}, \tag{5.7}$$

$$\hbar \frac{\partial \bar{\psi}}{\partial t} = \frac{\hbar}{2(x-\bar{q})} \frac{\partial \bar{\psi}}{\partial x} - \Big(\frac{\bar{p} + \bar{q}^2 + t/2}{2(x-\bar{q})} - \frac{\bar{q}}{2} \Big) \bar{\psi}, \tag{5.8}$$

associated with the second Painlevé equation

$$\bar{P}_2 \ : \ \hbar^2 \frac{d^2 \bar{q}}{dt^2} = 2\bar{q}^3 + t\bar{q} - \theta.$$

Here

$$\bar{f} = \hbar \frac{1}{x-\bar{q}}, \quad \bar{g} = x^4 + tx^2 - 2\theta x + 2\bar{H} + \frac{t^2}{4} - \hbar \bar{q} - \hbar \frac{\bar{p} + \bar{q}^2 + t/2}{x - \bar{q}},$$

and $\bar{H}$ is the Hamiltonian for $\bar{P}_2$ given by

$$\bar{H} = \bar{H}(\bar{q}, \bar{p}, t) = \frac{1}{2}\bar{p}^2 + \left(\bar{q}^2 + \frac{t}{2} \right) \bar{p} + \left(\theta + \frac{\hbar}{2} \right) \bar{q}. \tag{5.9}$$

As well as Theorem 3.2, the quantum curve is also related to the second Painlevé equation, but with a parameter shift $\theta \mapsto \theta + \hbar/2$. Again we emphasize that we regard $\bar{q}$ in the coefficients of isomonodromy system as the formal power series solution $\bar{q}(t,\hbar) = \sum_{m=0}^{\infty} \hbar^m \bar{q}_m(t)$ of $\bar{P}_2$. Obviously, the top term of the formal solution $q(t,\hbar)$ of P_2 and that of $\bar{q}(t,\hbar)$ are the same: $q_0(t) = \bar{q}_0(t)$. Hence the equations (5.7) and (2.7) have the same semi-classical limit (2.19). These facts imply that the quantum curve depends on how we define $F_{g,n}$ by integrating the Eynard-Orantin differential $W_{g,n}$. This is our new observation that cannot be seen in the case of the first Painlevé equation.

Remark 5.2. As we will see below, the formal series $a(t,\hbar)$ can also be defined (up to constant) by the condition

$$\frac{\partial a}{\partial t}(t,\hbar) = [\bar{\sigma}(t,\hbar) - \sigma(t,\hbar)]_{\hbar^{\geq 2}}.$$

Here $\sigma(t,\hbar)$ is defined in terms the solution of P_2 (not of $\bar{P}_2$) as in (2.9), and

$$\bar{\sigma}(t,\hbar) = \bar{H}(\bar{q}(t,\hbar), \bar{p}(t,\hbar), t).$$

Also, for a formal series $c(\hbar) = \sum_{k=0}^{\infty} \hbar^k c_k$, we have used the symbol $[c(\hbar)]_{\hbar^{\geq 2}}$ for $\sum_{k=2}^{\infty} \hbar^k c_k$.

The rest of this section is devoted to a proof of Theorem 5.1. It will be proved similarly to Theorem 3.2.

Let $\bar{R}^{(\pm)} = \sum_{m=0}^{\infty} \hbar^{m-1} \bar{R}_m^{(\pm)}(x,t)$ be the formal solution of the Riccati equation

$$\hbar^2 \left(\bar{R}^2 + \frac{d\bar{R}}{dt} \right) = \hbar \bar{f} \bar{R} + \bar{g} \tag{5.10}$$
$$= \hbar \left(\frac{\hbar \bar{R} - (\bar{p} + \bar{q}^2 + t/2)}{x - \bar{q}} - \bar{q} \right) + x^4 + tx^2 - 2\theta x + 2\bar{H} + \frac{t^2}{4}$$

associated with (5.7). The first two terms are given explicitly as follows:

$$\bar{R}_0^{(\pm)}(x,t) = \pm (x - q_0) \sqrt{r(x,t)}, \quad \bar{R}_1^{(\pm)}(x,t) = -\frac{x + q_0}{2r(x,t)}.$$

As well as (2.24), $\bar{R}$ satisfies

$$\hbar \frac{\partial \bar{R}^{(\pm)}}{\partial t} = \frac{\partial}{\partial x} \left(\frac{\hbar \bar{R}^{(\pm)} - (\bar{p} + \bar{q}^2 + t/2)}{2(x - \bar{q})} \right). \tag{5.11}$$

Using the equality, we obtain

Proposition 5.3 (C.f., [14, 15]). *The formal series*

$$\bar{\psi}_+(x,t,\hbar) = \exp \left(\frac{1}{\hbar} \bar{S}_0(x,t) + \bar{S}_1(x,t) + \sum_{m=2}^{\infty} \hbar^{m-1} \int_{\infty^{(+)}}^{x} \bar{R}_m^{(+)}(x,t) dx \right) \tag{5.12}$$

satisfies both (5.7) *and* (5.8).

Proposition 5.3 can be proved by a similarly method used in the proof of Theorem 2.5.

Let us compare (5.12) to $\bar{\psi}$ given by (5.6). We can check that (the logarithm of) these formal series agree up to $\hbar^2$ by direct computation. To prove the coincidence of (5.6) and (5.12) in full order, we will use

Proposition 5.4. *For each $m \geq 2$, we obtain*

$$\sum_{\substack{a+b=m+1 \\ a,b \geq 0}} \frac{\partial \bar{S}_a}{\partial x} \frac{\partial \bar{S}_b}{\partial x} + \frac{\partial^2 \bar{S}_m}{\partial x^2} - 2 \frac{\partial \bar{S}_m}{\partial t} = 2[\sigma]_{\hbar^{m+1}}. \tag{5.13}$$

The equality (5.13) is the counterpart of (4.14), which can be proved in the same way as a consequence of the topological recursion. Theorem 5.1 follows from Proposition 5.3, Proposition 5.4 and the following statements.

Proposition 5.5. *Let $m \geq 2$ be an integer.*

(i) *The function $a_m(t)$ defined by*

$$a_m(t) = \lim_{x \to \infty} \bar{S}_m(x,t)$$

satisfies

$$\frac{\partial a_m}{\partial t}(t) = [\bar{\sigma}(t,\hbar) - \sigma(t,\hbar)]_{\hbar^{m+1}}. \tag{5.14}$$

(ii) *The function $a_m(t)$ also satisfies*

$$\bar{S}_m(x,t) = \int_{\infty^{(+)}}^{x} \bar{R}_m^{(+)}(x,t)dx + a_m(t). \tag{5.15}$$

Proof. First, we note that

$$\bar{\sigma}(t,\hbar) = \sigma(t,\hbar)\big|_{\theta\mapsto\theta+\hbar/2}$$

holds. This follows from the fact that $(\bar{q}(t,\hbar),\bar{p}(t,\hbar))$ and $(q(t,\hbar),p(t,\hbar))|_{\theta\mapsto\theta+\hbar/2}$ satisfy the same Hamiltonian system having the unique formal power series solution. Using (3.16), we have

$$\begin{aligned}[\bar{\sigma}(t,\hbar) - \sigma(t,\hbar)]_{\hbar^{m+1}} &= \left[\sum_{n\ge1} \frac{\hbar^n}{n!2^n}\frac{\partial^n}{\partial\theta^n}\left(\sum_{g\ge0}\hbar^{2g}\frac{\partial F_g}{\partial t}\right)\right]_{\hbar^{m+1}} \\ &= \frac{\partial}{\partial t}\left(\sum_{\substack{2g+n=m+1\\ g\ge0,n\ge1}} \frac{1}{n!2^n}\frac{\partial^n F_g}{\partial\theta^n}\right).\end{aligned}$$

On the other hand, it follows from the definitions of $\bar{S}_m$ and $\bar{F}_{g,n}$ that

$$\lim_{x\to\infty}\bar{S}_m = \lim_{z\to\infty}\sum_{\substack{2g-2+n=m-1\\ g\ge0,n\ge1}}\frac{\bar{F}_{g,n}(z,\dots,z)}{n!} = \sum_{\substack{2g-2+n=m-1\\ g\ge0,n\ge1}}\frac{1}{n!2^n}\int_0^\infty\cdots\int_0^\infty W_{g,n}$$

holds. Then, the equality (5.14) follows from the variation formula (3.22) and (3.23) with respect to θ.

Next we prove (5.15) by the induction. Suppose that we have given $k \ge 2$ and (5.15) holds for $m = 0,1,\dots,k$. In particular the assumption implies

$$\frac{\partial\bar{S}_m}{\partial x} = \bar{R}_m, \quad m = 0,1,\dots,k. \tag{5.16}$$

Applying by $\partial/\partial t$ to both hand-sides of (5.15) for $m = k$, we have

$$\frac{\partial\bar{S}_k}{\partial t} = \left[\frac{\hbar\bar{R} - (\bar{p}+\bar{q}^2+t/2)}{2(x-\bar{q})} - \frac{\bar{q}}{2}\right]_{\hbar^k} + [\bar{\sigma}-\sigma]_{\hbar^{k+1}}$$

through a similar computation presented in the proof of Theorem 2.5. Then, under the induction hypothesis (c.f., (5.16)), the equality (5.13) shows that $\partial S_{k+1}/\partial x$ and $\bar{R}_{k+1}$ satisfies the same recursion relation. Hence the uniqueness of the solution implies

$$\frac{\partial \bar{S}_{k+1}}{\partial x}(x,t) = \bar{R}_{k+1}(x,t). \tag{5.17}$$

Integrating the both hands-sides of (5.17) with respect to x from ∞, we have (5.15) for $m = k+1$. □

References

[1] Aoki T., Kawai T. and Takei Y., WKB analysis of Painlevé transcendents with a large parameter II, in *Structure of Solutions of Differential Equations*, World Scientific, 1996, pp.1–49.

[2] Bergére M. and Eynard B., Determinantal formulae and loop equations, arXiv:0901.3273 [math-ph].

[3] Borot G. and Eynard B., Tracy-Widom GUE law and symplectic invariants, arXiv:1011.1418 [nlin.SI].

[4] ———, Geometry of spectral curves and all order dispersive integrable system, *SIGMA* **8** (2012), 100; arXiv:1110.4936 [math-ph].

[5] Bouchard V. and Eynard B., Reconstructing WKB from topological recursion, arXiv:1606.04498 [math-ph].

[6] Dumitrescu O. and Mulase M., Quantum curves for Hitchin fibrations and the Eynard-Orantin theory, *Lett. Math. Phys.* **104** (2014), 635–671; arXiv:1310.6022 [math.AG].

[7] ———, Quantization of spectral curves for meromorphic Higgs bundles through topological recursion, arXiv:1411.1023 [math.AG].

[8] ———, Lectures on the topological recursion for Higgs bundles and quantum curves, arXiv:1509.09007 [math.AG].

[9] Dunin-Barkowski P., Mulase M., Norbury P., Popolitov A. and Shadrin S., Quantum spectral curve for the Gromov-Witten theory of the complex projective line, *J. Reine Angew. Math.*; arXiv:1312.5336 [math-ph].

[10] Eynard B., Counting Surfaces, Progress in Mathematical Physics Volume **70**, Birkhäuser Mathematics, 2016.

[11] Eynard B. and Orantin N., Invariants of algebraic curves and topological expansion, *Comm. Number Theory Phys.* **1** (2007), 347–452; arXiv:math-ph/0702045.

[12] Fokas A.S., Its A.R., Kapaev A.A. and Novokshenov V.Y., Painlevé Transcendents: The Riemann-Hilbert Approach, Mathematical Surveys and Monographs, **128**, AMS, Providence, RI, 2006.

[13] Gukov S. and Sułkowski P., A-polynomial, B-model, and quantization, *JHEP*, (2012) 070; arXiv:1108.0002 [hep-th].

[14] Iwaki K., Parametric Stokes phenomena and Voros coefficients of the second Painlevé equation, *RIMS Kôkyûroku Bessatsu*, **B40** (2013), 221-242.

[15] ———, Parametric Stokes phenomenon for the second Painlevé equation, *Funk. Ekvac.*, **57** (2014), 173–243.

[16] ———, On WKB theoretic transformations for Painlevé transcendents on degenerate Stokes segments, *Publ. RIMS.*, **51** (2015), 1–57.
[17] Iwaki K. and Marchal O., Painlevé 2 equation with arbitrary monodromy parameter, topological recursion and determinantal formulas, arXiv:1411.0875 [math-ph].
[18] Iwaki K., Marchal O. and Saenz A., Painlevé equations, topological type property and reconstruction by the topological recursion, arXiv:1601.02517 [math-ph].
[19] Iwaki K. and Saenz A., Quantum curve and the first Painlevé equation, *SIGMA* **12** (2016), 011; arXiv:1507.06557 [math-ph].
[20] Jimbo. M and Miwa T., Monodromy perserving deformation of linear ordinary differential equations with rational coefficients. II, *Physica D*, **2** (1981), 407–448.
[21] Jimbo M., Miwa T. and Ueno K., Monodromy preserving deformations of linear ordinary differential equations with rational coefficients I, General theory and tau function, *Physica D*, **2** (1981), 306–352.
[22] Kamimoto S. and Koike T., On the Borel summability of 0-parameter solutions of non-linear ordinary differential equations, preprint of RIMS-1747 (2012).
[23] Kawai T. and Takei Y., WKB analysis of Painlevé transcendents with a large parameter. I, *Adv. Math.*, **118** (1996), 1–33.
[24] ———, WKB analysis of Painlevé transcendents with a large parameter. III. Local reduction of 2-parameter Painlveé transcendents, *Adv. Math.*, **134** (1998), 178–218.
[25] ———, Algebraic Analysis of Singular Perturbation Theory, Translations of Mathematical Monographs **227**, AMS, 2005 (Japanese ver. 1998).
[26] Manabe M. and Sułkowski P., Quantum curves and conformal field theory, arXiv:1512.05785 [hep-th].
[27] Okamoto K., Polynomial Hamiltonians associated with Painlevé equations I, *Proc. Japan Acad. Ser. A Math. Sci.*, **56** (1980), 264–268.
[28] Painlevé P., Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme, *Acta Math.*, **25** (1902), 1–85.
[29] Takei Y., An explicit description of the connection formula for the first Painlevé equation, In *Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear*, Kyoto Univ. Press, 2000, pp. 271–296.